Technical Comments

Comments on "Rates of Change of Eigenvalues and Eigenvectors"

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IN a recent Note, R. L. Fox and M. P. Kapoor[1] presented expressions for the rates of change of eigenvalues and eigenvectors of a structure with respect to the design parameters. Although there is complete agreement here, with the authors, about the usefulness of this type of expressions in frequency optimization of structures, two comments seem necessary.

The first one is that such formulas have been known, for the $(A - \lambda I)X$ problem, for more than ten years. Bellman[2] obtained expressions for first-order perturbation of eigenvalues similar to Zarghamee's[3] formula, and first-order perturbations of eigenvectors closely related to formulation 2 given by the authors.[1] Bellman gives also recursive formulas for higher-order perturbations.

The second point is that the use of perturbations or of rates of change can give a poor estimate of the effects of finite changes in the design parameters. An example can be given for the case in which the variation matrix is very sparse, since exact expressions are available for finite changes. Lax[4] derived these formula starting from

$$(A - \lambda I)X = 0 \tag{1}$$

Calling $A^1 = A + B$ the new matrix, and X^1 and λ^1 any eigenvector and the corresponding eigenvalue, partitioning B and X^1 as

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad X^1 = \begin{Bmatrix} X_1{}^1 \\ X_2{}^1 \end{Bmatrix}$$

Lax obtains

$$X^1 = (\lambda^1 I - A)^{-1} B_1 X_1{}^1 \tag{2}$$

It can be shown that

$$C = (\lambda^1 I - A)^{-1} = \bar{X}(\lambda^1 I - \Lambda)\bar{X}'$$

where $\bar{X}$ is the orthonormal matrix $[X_1, \ldots, X_n]$ and Λ is the diagonal matrix $[\lambda_1]$ of the original problem (1). For $B_1 = b_1$ scalar

$$c_{11} = \sum_{i=1}^{n} \frac{(x_i{}^1)^2}{\lambda^1 - \lambda_i} \tag{3}$$

Introducing (3) into (2) and eliminating $(x_1{}^1)$ from both sides of (2) one obtains, for instance, for the first frequency $\lambda_1{}^1$, the polynomial equation

$$\frac{1}{b_1} = \sum_{i=1}^{n} \frac{(x_i{}^1)^2}{\lambda_1{}^1 - \lambda_i} \tag{4}$$

The perturbation formula[2] would give, for the same problem

$$1/b_1 = (x_1{}^2)/(\lambda_1{}^1 - \lambda_1) \tag{4a}$$

It is clear that, if $\lambda_1{}^1 \approx \lambda_2$, (4a) can lead to a very poor estimate of $\lambda_1{}^1$, as compared with the value given by (4). The interpretation of (2) in structural terms derives quite simply from the possibility of reducing the $(K - \lambda M)X = 0$ problem to (1), under mildly restrictive conditions. The variation matrix B can be interpreted as the variation of the original matrix A due to finite changes only in a few design parameters. In particular, (4) becomes the equation for the modified first frequency, after a finite change has occurred in one of the nonstructural masses applied to a joint.

The previous example indicates that the rates of change, though easy to obtain, might not be very convenient to use for frequency optimization. In fact, the accuracy in predicting the response to finite changes in the design parameters affects directly the performance of the optimization algorithm. This is true particularly of gradient methods, which have shown a tendency to "oscillate" in the neighborhood of a local optimum.

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References

[1] Fox, R. L. and Kapoor, M. P., "Rates of Change of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 6, No. 12, Dec. 1968, pp. 2426–2429.
[2] Bellman, R., *Introduction to Matrix Analysis*, McGraw-Hill, New York, 1960, pp. 61–63.
[3] Zarghamee, M. S., "Optimum Frequency of Structures," *AIAA Journal*, Vol. 6, No. 4, April 1968, pp. 749–750.
[4] Lax, M., "Localized Perturbations," *The Physical Review*, Vol. 94, 1954, pp. 1391.

Reply by Authors to Aldo Cella

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THE basic purpose of the original Note was to inform the readership, which we believe was largely unfamiliar with the subject expressions, and to suggest a different practical application for them. As Mr. Cella points out (see preceding Comment) some of the expressions have been available from other sources and some of these were also mentioned in our Note. As for Mr. Cella's comment that the extrapolations would provide "poor estimates of the effects of finite changes in the design parameters," a limited amount of testing with the formulas has shown this defect not to be the case in practice. This is especially true if the Rayleigh quotient form of approximation is used.

Perhaps some of the discrepancy in our conclusions involves our interpretation of finite changes and our application of the method. We are studying weight minimization of structure using the method of feasible directions and the changes in design for each redesign iteration are often quite modest. Furthermore, we are not attempting to raise the frequency at each step but rather lower the weight without violating any inequality constraints on frequency and dynamic stress and displacement. Perhaps it is the fact that mode shapes change only a limited amount as the design is changed, while some of the frequencies may change considerably, that gives the formulas their relative accuracy.

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